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Solution of flat crack problem by using variational principle and differential–integral equation

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Abstract

Variational principle is used to solve some flat crack problems in three-dimensional elasticity. In the formulation, the strain energy is evaluated by multiplying the crack opening displacement (COD) by the boundary traction. The boundary traction is related to the COD function by a differential–integral representation. By using an integration by part, the portion of the strain energy of the potential functional can be expressed by a repeated integral. In the integral all the integrated functions are non-singular. Letting the functional be minimum, the solution is obtained. In the actual solution, the COD function is represented by a shape function family in which several undetermined coefficients are involved. Using the variational principle, the coefficients are obtained. Several numerical examples are given with the stress intensity factors calculated along the crack border.

Keywords: Variational principle; Three-dimensional crack problem; Stress intensity factor

1. Introduction

In practice, actual flaws in three-dimensional structures are often approximated by a flat shape crack. Also, the flat crack problem belongs to a particular problem of the three-dimensional elasticity; and the relevant analysis possesses some difficulty. For this reason, the flat crack problem has been paid considerable attention to Sneddon and Lowengrub (1969) and Kassir and Sih (1975).

For the flat crack case, the hypersingular integral equations were formulated by many researchers (Bueckner, 1977; Ioakimidis, 1982; Lin'kov and Mogilevskaya, 1986). In the formulation, the integral should be understood in the sense of the finite part integral (Hadamards, 1923; Kaya and Erdogan, 1987). Since the hypersingular integral should be defined in a rigorous manner, one should investigate an appropriate integral rule in derivation and computation. Generally, the solution is more difficult by using this method. Also, the relevant differential–integral equations for the flat crack problem were suggested

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(Ioakimidis, 1982; Fabrikant, 1987; Chen et al., 1996, 1997). Because of the involved differential operator, it is not easy to use the equation for solving the problem numerically.

On the other hand, instead of using the differential or integral equation the variational principle in elasticity is an effect way to solve the elasticity problem. For the plane elasticity crack problem, the variational principle is used to solve the crack problem of a finite plate (Chen, 1983). Recently, a variational boundary integral method is developed for the analysis of three-dimensional cracks (Xu, 2000). The method is based on modeling the crack as a continuous distribution of dislocation loops.

In this paper, it is found that the conjunction of the variational principle and the differential–integral equation in flat crack problem is an effective way in this field. In the formulation, the strain energy is evaluated by multiplying the crack opening displacement (COD) by the boundary traction. The boundary traction is related to the COD function by a differential–integral representation. By using an integration by part, the portion of the strain energy of the potential functional can be expressed by a repeated integral. In the integral all the integrated functions are non-singular. Letting the functional be minimum, the solution is obtained. In the actual solution, the COD function is represented by a shape function family in which several undetermined coefficients are involved. Using the variational principle, the coefficients are obtained. In this paper, several flat crack problems are present with the calculated results. The problems include: (a) an elliptic crack (b) the interaction of two elliptic cracks and (c) a rectangular crack. The obtained results are compared with the previous solutions.

It is well known that it is not easy to use the standard finite method to the case of an infinite body. Secondly, if the hypersingular integral boundary element method is used, there are some difficult points. For example, the shape functions should take different forms for the inner element and the near boundary element. This will make the solution more difficult.

2. Analysis

Let us consider a region S that represents the flat crack (Fig. 1). Assume that the tractions applied on the upper and the lower crack faces are the same in magnitude and opposite in direction. The results to evaluate the traction from the COD function $W(x,y)$ are as follows (Chen et al., 1996)

$$\sigma_z = \frac{1}{H}\Delta_0 \int\int_S \frac{1}{r} W(x,y)\,\mathrm{d}x\,\mathrm{d}y \tag{1}$$

$$\sigma_z = \frac{1}{H}\,\text{v.f.} \int\int_S \frac{1}{r^3} W(x,y)\,\mathrm{d}x\,\mathrm{d}y \tag{2}$$

where S denotes the region occupied by the flat crack, and

$$r^2 = (x-x_0)^2 + (y-y_0)^2, \quad \Delta_0 = \frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial y_0^2}, \quad H = \frac{2\pi(1-v)}{G} \tag{3}$$

with G being the shear modulus of elasticity, and v being the Poisson's ratio. In Eq. (2) v.f. means that the integral should be understood in the sense of the finite part integral.

Let the boundary condition take the form

$$\sigma_z = \overline{\sigma}_z = -qQ(x_0,y_0), \quad (x_0,y_0) \in S \tag{4}$$

where q is a constant traction. From Eqs. (1) and (4), the following differential–integral equation is obtained,

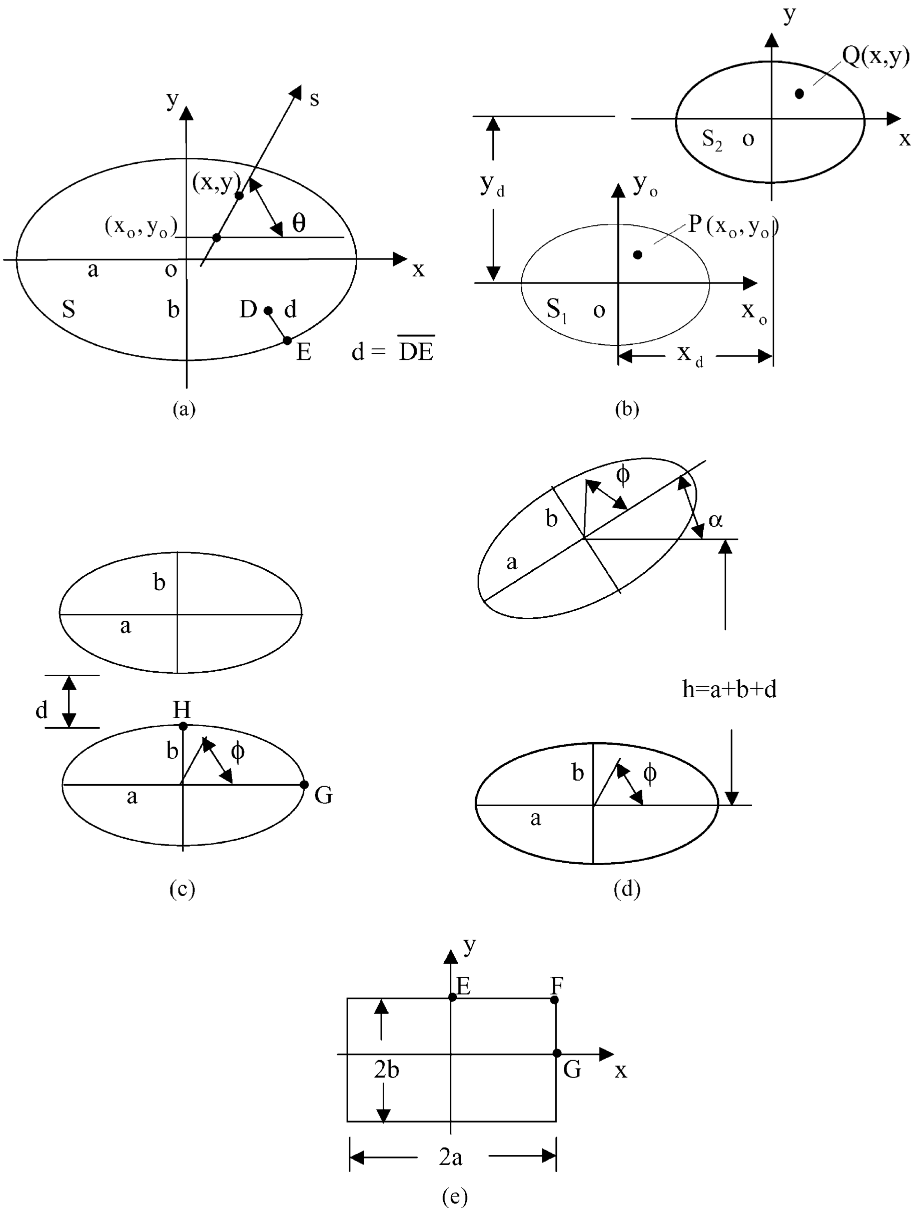

Fig. 1. (a) An elliptical crack, (b) two elliptical cracks in parallel position, (c) two elliptical cracks in series, (d) two cracks with one fixed and other in rotation position and (e) a flat rectangular crack.

$$\Delta_0 \int_S \int \frac{1}{r} W(x,y)\,\mathrm{d}x\,\mathrm{d}y = -HqQ(x_0,y_0) \tag{5}$$

Alternatively, From Eqs. (2) and (4), the following hypersingular integral equation is obtained,

$$\text{v.f.}\int_S \int \frac{1}{r^3} W(x,y)\,\mathrm{d}x\,\mathrm{d}y = -HqQ(x_0,y_0) \tag{6}$$

In the following analysis, the least potential energy principle is used to find a final solution. It is known that in the present case the potential functional takes the form,

$$\Pi = \Pi_1 + \Pi_2 \tag{7}$$

$$\Pi_1 = \frac{1}{2}\int_S\int (-\sigma_z) W(x_0, y_0)\, \mathrm{d}x_0\, \mathrm{d}y_0 \tag{8}$$

$$\Pi_2 = -\int_S\int (-\overline{\sigma}_z) W(x_0, y_0)\, \mathrm{d}x_0\, \mathrm{d}y_0 = -q\int_S\int Q(x_0, y_0) W(x_0, y_0)\, \mathrm{d}x_0\, \mathrm{d}y_0 \tag{9}$$

where Π_1 is equal to the potential energy stored in body, and Π_2 is the potential energy from the applied loading. Since the normal for the upper crack face is always directed towards down, the minus (−) is placed before σ_z or $\overline{\sigma}_z$.

From above equations we see that the evaluation of the integral Π_2 is straightforward. For the integral Π_1, there are two ways to substitute the component σ_z. One is to use Eq. (1) and the other is to use Eq. (2). The following derivation will prove that the former has a particular advantage. Clearly, after using Eq. (1) we prefer to write the Π_1 in the form,

$$\Pi_1 = \frac{1}{2H}(P_{1x} + P_{1y}) \tag{10}$$

$$P_{1x} = -\int_S\int W(x_0, y_0)\left\{\frac{\partial^2}{\partial x_0^2}\int_S\int \frac{1}{r} W(x, y)\, \mathrm{d}x\, \mathrm{d}y\right\}\mathrm{d}x_0\, \mathrm{d}y_0 \tag{11}$$

$$P_{1y} = -\int_S\int W(x_0, y_0)\left\{\frac{\partial^2}{\partial y_0^2}\int_S\int \frac{1}{r} W(x, y)\, \mathrm{d}x\, \mathrm{d}y\right\}\mathrm{d}x_0\, \mathrm{d}y_0 \tag{12}$$

In the following derivation we will change the form of P_{1x} and P_{1y} to make these integrals in a more convenient fashion for computation. For convenience, in derivation P_{1x} is rewritten in the form,

$$P_{1x} = -\int_S\int W(x_0, y_0)\frac{\partial g(x_0, y_0)}{\partial x_0}\, \mathrm{d}x_0\, \mathrm{d}y_0 \tag{13a}$$

$$g(x_0, y_0) = \frac{\partial}{\partial x_0}\int_S\int \frac{1}{r} W(x, y)\, \mathrm{d}x\, \mathrm{d}y \tag{13b}$$

Clearly, from Eq. (3) we have the following equalities,

$$\frac{\partial}{\partial x_0}\left(\frac{1}{r}\right) = -\frac{\partial}{\partial x}\left(\frac{1}{r}\right), \qquad \frac{\partial}{\partial y_0}\left(\frac{1}{r}\right) = -\frac{\partial}{\partial y}\left(\frac{1}{r}\right) \tag{14}$$

In addition, considering that the function $W(x, y)$ vanishes along the crack boundary and performing an integration by part, Eqs. (13a) and (13b) becomes in the form,

$$P_{1x} = \int_S\int \frac{\partial W(x_0, y_0)}{\partial x_0} g(x_0, y_0)\, \mathrm{d}x_0\, \mathrm{d}y_0 \tag{15a}$$

$$g(x_0, y_0) = -\int_S\int \frac{\partial}{\partial x}\left(\frac{1}{r}\right) W(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_S\int \frac{\partial W(x, y)}{\partial x}\left(\frac{1}{r}\right)\mathrm{d}x\,\mathrm{d}y \tag{15b}$$

Finally, we have

$$P_{1x} = \int_S\int \frac{\partial W(x_0, y_0)}{\partial x_0}\left\{\int_S\int \frac{\partial W(x, y)}{\partial x}\frac{1}{r}\,\mathrm{d}x\,\mathrm{d}y\right\}\mathrm{d}x_0\,\mathrm{d}y_0 \tag{16}$$

Similarly,

$$P_{1y} = \int_S\int \frac{\partial W(x_0, y_0)}{\partial y_0}\left\{\int_S\int \frac{\partial W(x, y)}{\partial y}\frac{1}{r}\,\mathrm{d}x\,\mathrm{d}y\right\}\mathrm{d}x_0\,\mathrm{d}y_0 \tag{17}$$

From general analysis in fracture mechanics, we see that the COD function $W(x, y)$ at the point D has an estimation $\mathrm{O}(d^{1/2})$ (Fig. 1(a)), where $d = \overline{\mathrm{DE}}$ is the distance from a point "D" to the crack border point "E" along the normal of contour. This means that $\partial w(x, y)/\partial x$ and $\partial w(x, y)/\partial y$ at the point "D" have an estimation $\mathrm{O}(d^{-1/2})$. Also, if we use the polar coordinate (s, θ) with center at the point (x_0, y_0), the term $\mathrm{d}x\,\mathrm{d}y/r$ is changed into $\mathrm{d}s\,\mathrm{d}\theta$. Therefore, there is no singular integral involved in Eqs. (16) and (17).

In the solution, we may assume the COD function in the form of a shape function family,

$$W(x, y) = \sum_{j=1}^{M} X_j W_j(x, y) \tag{18}$$

where the $W_j(x, y)$ $(j = 1, 2, \ldots, M)$ are the COD shape functions, which can be assumed for individual case, and X_j $(j = 1, 2, \ldots, M)$ are the undetermined coefficients.

Substituting Eq. (18) into Eqs. (9), (10), (16) and (17), the potential functional can be expressed as

$$\Pi = \frac{1}{2}\sum_{j=1}^{M}\sum_{k=1}^{M} A_{jk} X_j X_k - \sum_{j=1}^{M} R_j X_j \tag{19}$$

where

$$\begin{aligned} A_{jk} = A_{kj} = {} & \frac{1}{H}\int_S\int \frac{\partial W_j(x_0, y_0)}{\partial x_0}\left\{\int_S\int \frac{\partial W_k(x, y)}{\partial x}\frac{1}{r}\,\mathrm{d}x\,\mathrm{d}y\right\}\mathrm{d}x_0\,\mathrm{d}y_0 \\ & + \frac{1}{H}\int_S\int \frac{\partial W_j(x_0, y_0)}{\partial y_0}\left\{\int_S\int \frac{\partial W_k(x, y)}{\partial y}\frac{1}{r}\,\mathrm{d}x\,\mathrm{d}y\right\}\mathrm{d}x_0\,\mathrm{d}y_0 \quad (j, k = 1, 2, \ldots, M) \end{aligned} \tag{20}$$

$$R_j = \int_S\int Q(x, y) W_j(x, y)\,\mathrm{d}x\,\mathrm{d}y \quad (j = 1, 2, \ldots, M) \tag{21}$$

From the following condition

$$\frac{\partial \Pi}{\partial X_j} = 0 \quad (j = 1, 2, \ldots, M) \tag{22}$$

we obtain the the following algebraic equation,

$$\sum_{k=1}^{M} A_{jk} X_k = R_j \quad (j = 1, 2, \ldots, M) \tag{23}$$

In Eq. (20) the relation $A_{jk} = A_{kj}$ can be easily proved by the Betti's reciprocal theorem in elasticity. After the solution for the COD function $W(x, y)$ is obtained, the stress intensity factor can be evaluated by (Chen et al., 1996)

$$K_1 = \frac{\pi\sqrt{2\pi}}{H} \lim_{d \to 0} d^{-1/2} W(x, y) \tag{24}$$

In case of two elliptical cracks, similar formulation can be completed (Fig. 1(b)). Assume that the two cracks have the same configuration. Thus, the same COD shape function can be used for two cracks. In addition to evaluate the elements A_{jk} of the matrix **A**, a matrix **B** with the elements B_{jk} is evaluated. The element B_{jk} is defined as

$$B_{jk} = B_{kj} = \frac{1}{H} \int\int_{S_1} \frac{\partial W_j(x_0, y_0)}{\partial x_0} \left\{ \int\int_{S_2} \frac{\partial W_k(x, y)}{\partial x} \frac{1}{r} \, \mathrm{d}x \, \mathrm{d}y \right\} \mathrm{d}x_0 \, \mathrm{d}y_0$$
$$+ \frac{1}{H} \int\int_{S_1} \frac{\partial W_j(x_0, y_o)}{\partial y_0} \left\{ \int\int_{S_2} \frac{\partial W_k(x, y)}{\partial y} \frac{1}{r} \, \mathrm{d}x \, \mathrm{d}y \right\} \mathrm{d}x_0 \, \mathrm{d}y_0 \quad (j, k = 1, 2, \ldots, M) \tag{25}$$

where S_1 and S_2 denote the first and second cracks, respectively and

$$r^2 = (x + x_d - x_0)^2 + (y + y_d - y_0)^2 \tag{26}$$

The notations x, x_d, x_0, y, y_d and y_0 are indicated in Fig. 1(b). Physically, the matrix **B** with the elements B_{jk} represents the mutual influence between two cracks. This can be seen from following results. In the repeat integral shown by Eq. (25), the function $W_j(x_0, y_0)$, $(x_0, y_0) \in S_1$ denotes the jth COD shape function defined for the crack in S_1, and the function $W_k(x, y)$, $(x, y) \in S_2$ denotes the kth COD shape function defined for the crack in S_2. Therefore, the component B_{jk} can be considered as an influence of jth COD shape function defined for the crack in S_1 on the kth COD shape function defined for the crack in S_2.

The algebraic equation for case of two cracks is formulated immediately in the form,

$$\begin{vmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}_s & \mathbf{A} \end{vmatrix} \begin{vmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{vmatrix} = \begin{vmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \end{vmatrix} \tag{27}$$

where $\mathbf{B}_s$ denotes a matrix which is symmetry with respect to the matrix **B**, $\mathbf{X}_1$ and $\mathbf{X}_2$ are two unknown vectors for two cracks which are involved in the COD shape functions. $\mathbf{R}_1$ and $\mathbf{R}_2$ are two vectors for two cracks which are derived from the given loading. After the solutions for the unknown vectors $\mathbf{X}_1$ and $\mathbf{X}_2$ are obtained, the COD function is also evaluated from Eq. (18). Furthermore, the SIF can be obtained by using Eq. (24).

3. Solution technique and numerical examples

To prove the efficiency of the suggested formulation, several numerical examples are given below. Some particular points in computation are also described in the examples. In all examples, the crack face is loaded by a constant pressure q, or $\sigma_z = \bar{\sigma}_z = -q$ in Eq. (4).

Example 3.1

As the first example, a flat crack is under consideration (Fig. 1(a)). By using the differential–integral equation method, the problem has a closed form solution in the form (Chen et al., 1996),

$$W(x,y) = \frac{Hqb}{2\pi E(k)}\left[1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2\right]^{1/2} \tag{28}$$

where

$$k = (1 - (b/a)^2)^{1/2} \tag{29}$$

a and b are major and minor axis in ellipse, respectively. $E(k)$ denotes the complete elliptical integral of second kind.

In the numerical solution, we choose

$$W(x,y) = \frac{Hbq}{2\pi E(k)}\left(1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2\right)^{1/2}\left[c_1 + c_2\left(\frac{x}{a}\right)^2 + c_3\left(\frac{y}{b}\right)^2 + c_4\left(\frac{x}{a}\right)^4 + c_5\left(\frac{x}{a}\right)^2\left(\frac{y}{b}\right)^2 + c_6\left(\frac{y}{b}\right)^4\right] \tag{30}$$

In the numerical integration of the integral shown in Eq. (20), the following procedures are used. We take one integral in Eq. (20) as an example, which may be rewritten in the following form,

$$I(x_0,y_0) = \int_S\int \frac{\partial W_k(x,y)}{\partial x}\frac{1}{r}\,\mathrm{d}x\,\mathrm{d}y \tag{31a}$$

$$g = \int_S\int \frac{\partial W_j(x_0,y_0)}{\partial x_0} I(x_0,y_0)\,\mathrm{d}x_0\,\mathrm{d}y_0 \tag{31b}$$

For evaluating the integral $I(x_0,y_0)$ which depends on the location of the point (x_0,y_0), it is suitable to perform the integration in a particular polar coordinate (s,θ) with the center at the point (x_0,y_0). Therefore, the component $\mathrm{d}x\,\mathrm{d}y/r$ is changed into $\mathrm{d}s\,\mathrm{d}\theta$. In s-direction, the Gaussian quadrature rule with 20 divisions is used. In θ-direction, the Simpson quadrature rule with 40 divisions is used. For evaluating the integral shown in Eq. (31b), the first step is to perform integration in y_0-direction with a fixed x_0 value, and the next step is in x_0-direction. In both steps, the Gaussian quadrature rule with 50 divisions is used.

For nine k values of $0, \sin(\pi/18), \ldots, \sin(8\pi/18)$, the solutions obtained for the coefficients c_i in Eq. (30) are listed in Table 1.

From the results, we see that the deviation of the numerical solution from the exact solution ($c_1 = 1$, $c_i = 0$, $i = 2,3,4,5,6$) is negligible.

Table 1
The calculated coefficients c_i $(i = 1,2,\ldots,6)$ in Eq. (30)

k	b/a	c_1	c_2	c_3	c_4	c_5	c_6
0	1.0000	0.9997	0.0028	0.0000	−0.0070	0.0028	−0.0002
$\sin(\pi/18)$	0.9848	0.9997	0.0028	0.0000	−0.0070	0.0028	−0.0002
$\sin(2\pi/18)$	0.9397	0.9997	0.0028	0.0000	−0.0069	0.0026	−0.0002
$\sin(3\pi/18)$	0.8660	0.9997	0.0028	0.0000	−0.0068	0.0024	−0.0001
$\sin(4\pi/18)$	0.7660	0.9998	0.0028	0.0000	−0.0066	0.0021	−0.0001
$\sin(5\pi/18)$	0.6428	0.9998	0.0027	−0.0001	−0.0062	0.0016	0.0000
$\sin(6\pi/18)$	0.5000	0.9998	0.0025	−0.0001	−0.0056	0.0011	0.0000
$\sin(7\pi/18)$	0.3420	0.9998	0.0021	0.0000	−0.0045	0.0005	0.0000
$\sin(8\pi/18)$	0.1736	1.0019	0.0013	0.0001	−0.0025	0.0002	−0.0001

Example 3.2

As the second example, two cracks with the elliptical configuration are in parallel (Fig. 1(c)). Considering the symmetry condition with respect to the x-coordinate, the COD shape function may be assumed in the form,

$$W(x,y) = \left(1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2\right)^{1/2} \left[c_1 + c_2\left(\frac{y}{b}\right) + c_3\left(\frac{x}{a}\right)^2 + c_4\left(\frac{y}{b}\right)^2 + c_5\left(\frac{x}{a}\right)^2\left(\frac{y}{b}\right) + c_6\left(\frac{y}{b}\right)^3 + c_7\left(\frac{x}{a}\right)^4 + c_8\left(\frac{x}{a}\right)^2\left(\frac{y}{b}\right)^2 + c_9\left(\frac{y}{b}\right)^4 + c_{10}\left(\frac{x}{a}\right)^4\left(\frac{y}{b}\right) + c_{11}\left(\frac{x}{a}\right)^2\left(\frac{y}{b}\right)^3 + c_{12}\left(\frac{y}{b}\right)^5 + c_{13}\left(\frac{x}{a}\right)^6 + c_{14}\left(\frac{x}{a}\right)^4\left(\frac{y}{b}\right)^2 + c_{15}\left(\frac{x}{a}\right)^2\left(\frac{y}{b}\right)^4 + c_{16}\left(\frac{y}{b}\right)^6\right] \tag{32}$$

The above-mentioned computation conditions are used in the present case. Two sets of numerical example are carried out. In the first set of the example, we choose $b/a = 0.25, 0.5, 0.75$ and 1, and $d/b = 2/9, 0.5, 1.2$ and 2.0. The calculated results at the point H are expressed by

$$K_1 = f(b/a, d/b)qF(a, b, \pi/2) \tag{33a}$$

$$F(a, b, \phi) = \frac{1}{E(k)}\sqrt{\frac{\pi b}{a}}(a^2 \sin^2\phi + b^2\cos^2\phi)^{1/4} \tag{33b}$$

where the function $f(b/a, d/b)$ represents the magnified factor of SIF. The value $qF(a, b, \phi)$ represents the stress intensity factor at a border point $(x = a\cos\phi, y = \sin\phi)$ of a single elliptical crack under uniform loading q.

The calculated results are listed in Table 2. Comparison of the results are also listed in Table 2. Coincidence with various sources of solutions has been found from Table 2.

In the second set of the example, we choose $b/a = 0.25, 0.5, 0.75$ and 1, and $d/b = 0.1, 0.2, 0.5$ and 1.0. The calculated results for the point $(x = a\cos\phi, b = b\sin\phi)$ along the curve GH are expressed by

$$K_1 = g(b/a, d/b, \phi)qF(a, b, \phi) \tag{34}$$

where the function $g(b/a, d/b, \phi)$ represents the magnified factor of SIF. The calculated results are shown in Fig. 2.

Example 3.3

As the third example, the lower crack is in a fixed position and the upper crack is subject to rotation, and the two cracks have the same elliptical configuration (Fig. 1(d)). In the case, we need to assume the following COD function,

Table 2
The calculated non-dimensional SIFs $f(b/a, d/b)$ (see Fig. 1(c) and Eqs. (33a) and (33b))

b/a	$2b/(d+2b)$			
	0.9 (2/9)	0.8 (0.50)	0.625 (1.20)	0.5 (2.00)
0.25	1.3550	1.1719	1.0598	1.0272
0.25 (Murakami, 1987)		1.181	1.063	1.028
0.50	1.2555	1.1174	1.0365	1.0153
0.50 (Murakami, 1987)		1.126	1.038	1.016
0.75	1.1954	1.0851	1.0246	1.0100
1.00	1.1538	1.0642	1.0178	1.0071
1.00 (Murakami, 1987)		1.068	1.018	1.007

Values in parentheses denote d/b.

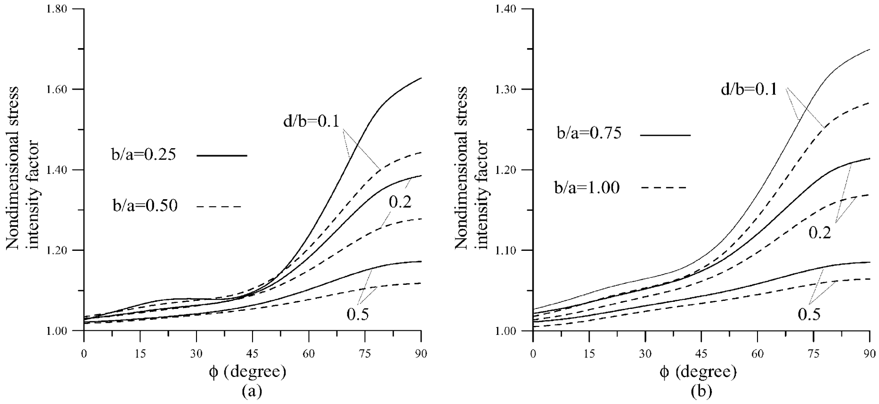

Fig. 2. Non-dimensional SIFs $g(b/a, d/a, \phi)$ for two elliptical cracks in series (a) $b/a = 0.25$ and 0.50, (b) $b/a = 0.75$ and 1.00.

$$
\begin{aligned}
W(x,y) = {} & \left(1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2\right)^{1/2} \left[c_1 + c_2\left(\frac{x}{a}\right) + c_3\left(\frac{y}{b}\right) + c_4\left(\frac{x}{a}\right)^2 + c_5\left(\frac{x}{a}\right)\left(\frac{y}{b}\right) + c_6\left(\frac{y}{b}\right)^2\right. \\
& + c_7\left(\frac{x}{a}\right)^3 + c_8\left(\frac{x}{a}\right)^2\left(\frac{y}{b}\right) + c_9\left(\frac{x}{a}\right)\left(\frac{y}{b}\right)^2 + c_{10}\left(\frac{y}{b}\right)^3 + c_{11}\left(\frac{x}{a}\right)^4 + c_{12}\left(\frac{x}{a}\right)^3\left(\frac{y}{b}\right) \\
& \left. + c_{13}\left(\frac{x}{a}\right)^2\left(\frac{y}{b}\right)^2 + c_{14}\left(\frac{x}{a}\right)\left(\frac{y}{b}\right)^3 + c_{15}\left(\frac{y}{b}\right)^4\right]
\end{aligned}
\tag{35}
$$

In the example, we choose $b/a = 0.5$ and $d/b = 0.2$ and the distance between centers of two cracks are assumed to be $h = a + b + d$. The inclined angle α of the upper crack is subject to change.

In this case, the calculated results for SIF at the boundary point ($x = a\cos\phi, b = b\sin\phi$) can be expressed as

$$K_1 = h_1(\alpha, \phi)qF(a, b, \phi) \quad \text{(for the upper crack)} \tag{36}$$

$$K_1 = h_2(\alpha, \phi)qF(a, b, \phi) \quad \text{(for the lower crack)} \tag{37}$$

where the functions $h_1(\alpha, \phi)$ and $h_2(\alpha, \phi)$ represents the magnified factor of SIF. The calculated results for three cases $\alpha = 0$, $\alpha = 5\pi/12$ and $\alpha = \pi/2$ are shown in Fig. 3. For $\alpha = 0$, $a = 2.0$, $b = 1.0$ and $d = 0.2$, the gap spacing is 1.2 ($= a + b + d - 2b$). In this case, the interaction effect is comparatively small. On contrary, for the $\alpha = \pi/2$, $a = 2.0$, $b = 1.0$ and $d = 0.2$, the gap spacing becomes 0.2 ($= a + b + d - a - b$). In this case, the interaction effect becomes larger. For example, for the upper crack, $h_1(\alpha, \phi)$ value reaches its maximum 1.2381 at the angle $\phi = 180°$. Similarly, for the lower crack $h_2(\alpha, \phi)$ value reaches its maximum 1.1168 at the angle $\phi = 90°$.

Example 3.4

As the fourth example, the crack has a rectangular configuration (Fig. 1(e)). In the case, we need to assume the following COD function in the form,

$$W(x,y) = \left(1 - \left(\frac{x}{a}\right)^2\right)^{1/2} \left(1 - \left(\frac{y}{b}\right)^2\right)^{1/2} \left[c_1 + c_2\left(\frac{x}{a}\right)^2 + c_3\left(\frac{y}{b}\right)^2 + c_4\left(\frac{x}{a}\right)^4 + c_5\left(\frac{x}{a}\right)^2\left(\frac{y}{b}\right)^2 + c_6\left(\frac{y}{b}\right)^4\right] \tag{38}$$

In computation, similar conditions are used in the present case. In the example, we choose $a/b = 1, 2$ and 4. The calculated results for SIF are expressed by

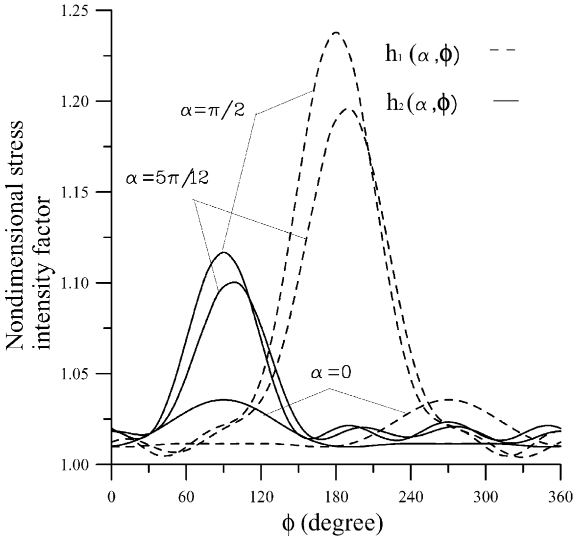

Fig. 3. Non-dimensional SIFs $h_1(\alpha, \phi)$ and $h_2(\alpha, \phi)$ for two elliptical cracks with one fixed and other in rotation position (see Fig. 1(d) and Eqs. (36) and (37)).

$$K_1 = f(a/b, x/a)q\sqrt{\pi b} \quad \text{(along the upper edge EF in Fig. 1(e))} \tag{39}$$

$$K_1 = g(a/b, y/b)q\sqrt{\pi b} \quad \text{(along the left edge GF in Fig. 1(e))} \tag{40}$$

where the function $f(a/b, x/a)$ and $g(a/b, y/b)$ represents the magnified factor of SIF. The calculated results are shown in Table 3 and Fig. 4. The results can be examined from the following consideration. Clearly, in the long strip case ($a/b \to \infty$), the $f(a/b, x/a)$ at the point E ($x/a = 0$) should be equal to 1. In the present example, when $a/b = 4$, the relevant value of the function $f(b/a, x/a)$ at $x/a = 0$ is 0.9636. Secondly, in the $a/b = 1$ case, the previous result for $f(b/a, x/a)$ at $x/a = 0$ is 0.76 (Murakami, 1987). In the present case, the relevant calculated value is 0.7616. Thus, the validity of computation is obtained indirectly.

Table 3
The calculated non-dimensional SIFs $f(a/b, x/a)$ and $g(a/b, y/b)$ for the rectangular flat crack (see Fig. 1(e) and Eqs. (39) and (40))

a/b	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Panel A										
1	0.7616	0.7602	0.7557	0.7475	0.7341	0.7135	0.6820	0.6337	0.5573	0.4261
2	0.9122	0.9113	0.9085	0.9029	0.8929	0.8757	0.8467	0.7978	0.7132	0.5557
4	0.9636	0.9634	0.9626	0.9612	0.9586	0.9530	0.9403	0.9107	0.8437	0.6863
Panel B										
1	0.7616	0.7602	0.7557	0.7475	0.7341	0.7135	0.6820	0.6337	0.5573	0.4261
2	0.8179	0.8151	0.8067	0.7921	0.7700	0.7389	0.6956	0.6352	0.5480	0.4103
4	0.8283	0.8248	0.8143	0.7961	0.7692	0.7321	0.6822	0.6152	0.5227	0.3845

Panel A: $x/a = 0, 0.1, \ldots, 0.9$; Panel B: $y/b = 0, 0.1, \ldots, 0.9$.

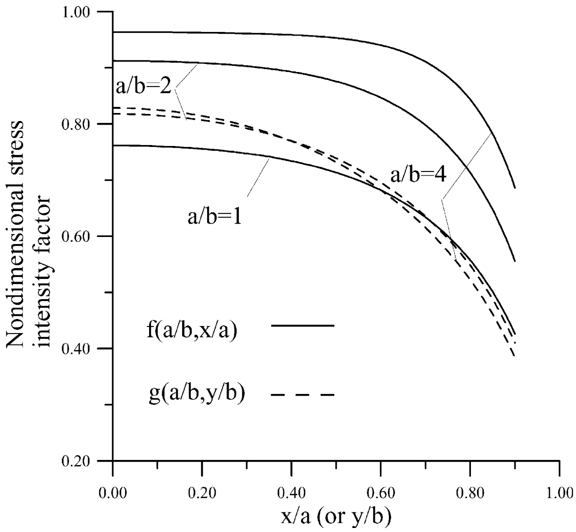

Fig. 4. Non-dimensional SIFs $f(a/b, x/a)$ and $g(a/b, y/b)$ for a flat rectangular crack (see Fig. 1(e) and Eqs. (39) and (40)).

4. Conclusions and discussions

In the present study, a variational principle in conjunction with the differential–integral equation is used to solve the flat crack problem. There is no singular integral involved in the formulation, which is a particular advantage of the suggested method. Also, the method can be used to the shear mode case in a flat crack problem. Meantime, on the base of the present formulation, the boundary element technique can also be developed to solve the flat crack problem. The mentioned extension will be investigated in a future study.

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